Volumes by Cylindrical Shells

For each problem, use the method of cylindrical shells to find the volume of the solid that results when the region enclosed by the curves is revolved about the the y-axis.

1) \( y = \sqrt{x} + 4 \)
\[ y = x^2 + 4 \]

\[
2\pi \int_{0}^{1} x(\sqrt{x} + 4 - (x^2 + 4)) \, dx
= \frac{3}{10} \pi
\]

2) \( y = 7 \)
\[ y = \sqrt{x} \]
\[ x = 0 \]
\[ x = 4 \]

\[
2\pi \int_{0}^{4} x(7 - \sqrt{x}) \, dx
= \frac{432}{5} \pi
\]

Critical thinking question:

3) Solve problem 2 using the method of washers. Why is this problem easier using cylindrical shells?

\[
\pi \int_{0}^{2} (y^2)^2 \, dy + \pi \int_{2}^{7} 4^2 \, dy = \frac{432}{5} \pi
\]
The cylindrical shell method requires one integral, while the disk method requires two.
For each problem, find the volume of the solid that results when the region enclosed by the curves is revolved about the the given axis. You may use the provided graph to sketch the curves and shade the enclosed region.

5) \( y = x^2 - 2 \)
   \( y = -2 \)
   \( x = 2 \)
   Axis: \( y = -2 \)

\[ \pi \int_{0}^{2} (x^2)^2 \, dx \]
\[ = \frac{32}{5} \pi = 20.106 \]

6) \( x = \sqrt{y} + 3 \)
   \( x = \frac{y}{2} + 3 \)
   Axis: \( x = 1 \)

\[ \pi \int_{0}^{4} \left( (\sqrt{y} + 2)^2 - \left( \frac{y}{2} + 2 \right)^2 \right) \, dy \]
\[ = 8\pi = 25.133 \]

Critical thinking questions:

7) Use the method of disks to derive the formula for the volume of a sphere of radius \( r \).

   \[ x^2 + y^2 = r^2, \ y = \sqrt{r^2 - x^2}, \ V = \pi \int_{-r}^{r} \left( \sqrt{r^2 - x^2} \right)^2 \, dx, \ V = \left( r^2x - \frac{x^3}{3} \right) \bigg|_{-r}^{r}, \ V = \frac{4}{3}\pi r^3 \]

8) A 6 cm diameter drill bit is used to drill a cylindrical hole through the middle of a sphere of radius 5 cm. What is the volume of the resulting object?

   \[ V = \pi \int_{-4}^{4} ((\sqrt{25 - x^2})^2 - 9) \, dx = \frac{256\pi}{3} \, cm^3 \]
Volumes of Revolution - Washers and Disks

For each problem, find the volume of the solid that results when the region enclosed by the curves is revolved about the the x-axis.

1) \( y = -x^2 + 1 \)
   \( y = 0 \)

\[
\pi \int_{-1}^{1} (-x^2 + 1)^2 \, dx
\]

\[
= \frac{16}{15} \pi \approx 3.351
\]

2) \( y = 2x + 2 \)
   \( y = x^2 + 2 \)

\[
\pi \int_{0}^{2} ((2x + 2)^2 - (x^2 + 2)^2) \, dx
\]

\[
= \frac{48}{5} \pi \approx 30.159
\]

For each problem, find the volume of the solid that results when the region enclosed by the curves is revolved about the given axis.

3) \( y = \sqrt{x} + 1 \)
   \( y = x^2 + 1 \)
   Axis: \( y = -1 \)

\[
\pi \int_{0}^{1} \left( (\sqrt{x} + 2)^2 - (x^2 + 2)^2 \right) \, dx
\]

\[
= \frac{49}{30} \pi \approx 5.131
\]

4) \( x = -y^2 + 2 \)
   \( x = y \)
   Axis: \( x = -2 \)

\[
\pi \int_{-2}^{1} \left( (\sqrt{x} + 2)^2 - (x^2 + 2)^2 \right) \, dy
\]

\[
= \frac{108}{5} \pi \approx 67.858
\]
For each problem, use the method of cylindrical shells to find the volume of the solid that results when the region enclosed by the curves is revolved about the y-axis. You may use the provided graph to sketch the curves and shade the enclosed region.

4) $y = 2x$
   \[ y = x^2 \]
   
   \[
   2\pi \int_{0}^{2} x(2x - x^2) \, dx
   \]
   
   \[ = \frac{8}{3} \pi \]

For each problem, use the method of cylindrical shells to find the volume of the solid that results when the region enclosed by the curves is revolved about the given axis. You may use the provided graph to sketch the curves and shade the enclosed region.

5) $y = -x^2 + 7$
   \[ y = x^2 + 5 \]
   Axis: $x = 2$

   \[
   2\pi \int_{-1}^{1} (2 - x)(-x^2 + 7 - (x^2 + 5)) \, dx
   \]
   
   \[ = \frac{32}{3} \pi \]
Area Between Curves

For each problem, find the area of the region enclosed by the curves.

1) \( y = 2x^2 - 8x + 10 \)  
   \( y = \frac{x^2}{2} - 2x - 1 \)
   
   \( x = 1 \)
   \( x = 3 \)

\[ \int_{1}^{3} \left( 2x^2 - 8x + 10 - \left( \frac{x^2}{2} - 2x - 1 \right) \right) \, dx \]
\[ = 11 \]

2) \( x = 2y^2 + 12y + 19 \)  
   \( x = -\frac{y^2}{2} - 4y - 10 \)
   
   \( y = -3 \)
   \( y = -2 \)

\[ \int_{-3}^{-2} \left( 2y^2 + 12y + 19 - \left( -\frac{y^2}{2} - 4y - 10 \right) \right) \, dy \]
\[ = \frac{29}{6} \approx 4.833 \]

3) \( y = \frac{x^2}{2} - 3x - \frac{1}{2} \)
   \( y = 3 \)

\[ \int_{-1}^{3} \left( 3 - \left( \frac{x^2}{2} - 3x - \frac{1}{2} \right) \right) \, dx \]
\[ = \frac{128}{3} \approx 42.667 \]

4) \( y = -\frac{x^3}{2} + 2x^2 \)
   \( y = -x^2 + 4x \)

\[ \int_{0}^{2} \left( -x^2 + 4x - \left( -\frac{x^3}{2} + 2x^2 \right) \right) \, dx + \int_{0}^{4} \left( \frac{x^3}{2} + 2x^2 - (-x^2 + 4x) \right) \, dx \]
\[ = 4 \]
For each problem, find the area of the region enclosed by the curves. You may use the provided graph to sketch the curves and shade the enclosed region.

5) \( y = -2x^2 - 1 \)
\[ y = -x + 3 \]
\[ x = 0 \]
\[ x = 1 \]
\[
\int_{0}^{1} \left( -x + 3 - (-2x^2 - 1) \right) \, dx
\]
\[
= \frac{25}{6} \approx 4.167
\]

6) \( y = 2\sqrt[3]{x^2} \)
\[ y = x \]
\[
\int_{0}^{8} (2\sqrt[3]{x^2} - x) \, dx
\]
\[
= \frac{32}{5} = 6.4
\]

7) \( y = -x^3 + 6x \)
\[ y = -x^2 \]
\[
\int_{-2}^{3} \left( -x^2 - (-x^3 + 6x) \right) \, dx +
\int_{-2}^{3} (-x^3 + 6x + x^2) \, dx
\]
\[
= \frac{253}{12} \approx 21.083
\]

8) \( y = -2 \cdot \sec^2 x \)
\[ y = 2\cos x \]
\[ x = 0 \]
\[ x = \frac{\pi}{4} \]
\[
\int_{0}^{\frac{\pi}{4}} \left( 2\cos x + 2 \cdot \sec^2 x \right) \, dx
\]
\[
= 2 + \sqrt{2} \approx 3.414
\]

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1. The region bounded by the graph of \( y = 2x - x^2 \) and the x-axis is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. What is the volume of the solid?

\[
\int_0^2 (2x-x^2)^2 \, dx = 1.33
\]

2. The region in Quadrant I bounded by the graph of \( f(x) = 1 - e^{-x} \) and \( g(x) = x^3 \) is the base of a solid. Find the volume of this solid, if

(a) For this solid, each cross section perpendicular to the x-axis is a semicircle

(b) For this solid, each cross section perpendicular to the x-axis is a square

\[
\text{Length is } x^3 - 1 + e^{-x} \quad \text{intersection } x^3 = 1 - e^{-x}
\]

\[
\int_0^1 \pi \left( \frac{x^3 - 1 + e^{-x}}{2} \right)^2 dx = 0.030
\]

\[
\int_0^1 \left( \frac{x^3 - 1 + e^{-x}}{2} \right)^2 dx = 0.039
\]

3. Let \( R \) be the region in Quadrant I bounded by the graph of \( y = e^x \), the y-axis, and the horizontal line \( y = 4 \).

(a) Find the area of \( R \). [Answer: 2.545]

(b) The region \( R \) is the base of a solid. For this solid, each cross section perpendicular to the y-axis is a square. Find the volume of this solid. [Answer: 2.597]
4. The base of a solid is the region bounded by the graph of \( y = 1 - x^3 \) and the \( x \)-axis. For this solid, each cross section perpendicular to the \( x \)-axis is a rectangle with height three times the base. What is the volume of this solid? [Answer: 3.2]

\[
\int_{-1}^{1} 3(1-x^2)^2 \, dx = 3.2
\]

5. Let \( R \) be the region bounded by the graph of \( y = \ln(x^2 + 1) \), the horizontal line \( y = 3 \), and the vertical line \( x = 1 \), as shown in the figure.
(a) Find the area of \( R \). [Answer: 3.310]
(b) The region \( R \) is the base of a solid. For this solid, each cross section perpendicular to the \( x \)-axis is an equilateral triangle. [Hint: Volume of an equilateral triangle = \( \frac{s^3 \sqrt{3}}{4} \)]
(c) Another solid whose base is also the region \( R \). For this solid, each cross section perpendicular to the \( x \)-axis is a semicircle with diameter across the base. Find the volume of this solid. [Answer: 1.854]

(a) \[
\int_{1}^{e^2-1} (3 - \ln(x^2+1)) \, dx = 3.31
\]

(b) \[
\sqrt{\frac{e^{3-1}}{2}} (3 - \ln(x^2+1))
\]

\[
\int_{1}^{\sqrt{\frac{e^{3-1}}{2}}} (3 - \ln(x^2+1))^2 \, dx = 2.045 \text{ double check this one!}
\]

(c) \[
\int_{1}^{e^3-1} \frac{\pi}{2} \left( \frac{3 - \ln(x^2+1)}{2} \right)^2 \, dx = 1.854
\]